European Journal of Operational Research 241 (2015) 830–841

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Decision Support

Modeling assignment-based pairwise comparisons within integrated framework for value-driven multiple criteria sorting

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ARTICLE INFO

Article history:
Received 13 January 2014
Accepted 19 September 2014
Available online 26 October 2014

Keywords:
Multiple criteria analysis
Sorting
Multi-Attribute Value Theory
Preference disaggregation
Robust Ordinal Regression

ABSTRACT

We introduce a new preference disaggregation modeling formulations for multiple criteria sorting with a set of additive value functions. The preference information supplied by the Decision Maker (DM) is composed of: (1) possibly imprecise assignment examples, (2) desired class cardinalities, and (3) assignment-based pairwise comparisons. The latter have the form of imprecise statements referring to the desired assignments for pairs of alternatives, but without specifying any concrete class. Additionally, we account for preferences concerning the shape of the marginal value functions and desired comprehensive values of alternatives assigned to a given class or class range. The exploitation of all value functions compatible with these preferences results in three types of results: (1) necessary and possible assignments, (2) extreme class cardinalities, and (3) necessary and possible assignment-based preference relations. These outputs correspond to different types of admitted preference information. By exhibiting different outcomes, we encourage the DM in various ways to enrich her/his preference information interactively. The applicability of the framework is demonstrated on data involving the classification of cities into liveability classes.

1. Introduction

Multiple criteria sorting (ordinal classification) involves the assignment of a set of alternatives evaluated using a set of criteria to one or several homogeneous classes. Despite being closely related to clustering (Meyer & Olteanu, 2013) and finding ordered segments (Chen, Cheng, & Hsu, 2013), this type of problem differs from both of them. On the one hand, clusters are not ordered, whereas classes are given in a preference order. On the other hand, segments do not need to be defined a priori, which is the case for classes. In any case, such discrimination among two or more ordered and pre-defined sets of alternatives is at the core of various real-world decision problems. Some recent sorting applications concern energy and electricity market (Diakoulaki, Zopounidis, Mavrotas, & Doumpos, 1999; Mavrotas, Diakoulaki, & Capros, 2003), climate change (Diakoulaki & Hontou, 2003), economy and finance (Doumpos & Zopounidis, 2011), stock portfolio selection (Xidonas, Mavrotas, & Psarras, 2009), cancer care (Belacel & Boulassel, 2000), airline market (Norese & Carbone, 2014), land-management (Macary, Almeida Dias, Figueira, & Roy, 2014), urban and territorial projects (Abastante, Bottero, Greco, & Lami, 2014), accreditation systems (Siskos, Grigoroudis, Krassadaki, & Matsatsinis, 2007), and tourism (Mailly, Abi-Zeid, & Pepin, 2014).

In this paper, the multiple criteria sorting model used to work out a recommendation is a set of value functions. Multi-Attribute Value Theory is a well established theory considering compensatory preference models that represent how DMs account for trade-offs among criteria. Such models are widely used and appreciated by the Multiple Criteria Decision Aiding community for their relatively small computational effort and easy interpretation. Using additive value functions requires specification of the parameters related to the formulation of marginal value functions. These parameters follow either directly or indirectly from preference information provided by the DM. The former involves direct specification of some parameter values. The latter concerns some examples of holistic or criterion-specific judgments, or requirements with respect to the delivered recommendation. This information is subsequently employed to induce values of the compatible preference model parameters which are able to restore the DM's exemplary judgments or requirements. Such indirect elicitation is usually called disaggregation.

In the last decades, methods that require indirect, imprecise, and incomplete preference statements of the DM are prevailing. In fact, several value-based disaggregation sorting methods have been already proposed in the literature. They require the DM to express her/his preferences by providing a set of assignment examples on a subset of alternatives (s)he knows

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http://dx.doi.org/10.1016/j.ejor.2014.09.050
0377-2217/

relatively well, called reference alternatives. When using such indirect preference information, there exist multiple (usually, infinitely many) compatible instances of the preference model. Various methods handle this ambiguity in different ways. Some of them, e.g., Bous, Fortemps, Glineur, and Pirlot (2010), Devaud, Groussaud, and Jacquet-Lagreze (1980), Doumpos and Zopounidis (2007), and Greco, Kadziński, and Słowiński (2011), select a single compatible value function, thus, providing precise assignment of alternatives. Other approaches, e.g., Greco, Mousseau, and Słowiński (2010), Kadziński and Tervonen (2013), and Köksalan and Bilgin Özpeynirci (2009), take into account all compatible value functions, and investigate the spaces of consensus and disagreement between recommendation suggested by these functions. These approaches are known under the name of Robust Ordinal Regression. The results of Robust Ordinal Regression are materialized with the possible and necessary assignments, that is sorting recommendations confirmed by all or at least one compatible value function, respectively. A recent study in Doumpos, Zopounidis, and Galariotis (2014) compares experimental results on the relationship between the outcomes of a single decision model (additive value function) and the ones from the whole set of compatible model instances.

The type of admitted preference information and elements of responses obtained by the DMs, have a great impact on the consistency between value system of the stakeholders, the evolution of the decision process and recommendation of a specific decision. Nowadays, the types of admitted preference information, models, procedures, and provided results are more often perceived as a communication and reflection tool. In this spirit, the recent trend in Multiple Criteria Decision Aiding consists in accounting for types of preference information which have not received due attention in existing methods, as well as conducting diversified robustness analysis for the delivered results. Using new types of preference information increases the flexibility of the interactive procedure, thus enabling the consideration of any preference information coming from the DM. The latter aims at increasing the range of tools that can be used for looking more thoroughly into the problem, by exploring, interpreting, or testing scenarios. When it comes to recently proposed new types of preference information, let us recall desired class cardinalities, e.g., “we wish to accept at most 10 candidates” or “we need to reject at least 30 applications” (Kadziński & Słowiński, 2013; Mousseau, Dias, & Figueira, 2003), which now can be employed along with the traditionally used assignment examples.

As far as robustness analysis of sorting recommendation is concerned, apart from the already mentioned necessary and possible assignments, one has recently proposed to consider three types of results:

- assignment-based preference relations, which admit the comparison of a sorting recommendation for pairs of alternatives (Kadziński & Tervonen, 2013),
- class acceptability indices representing the shares of compatible preference model instances assigning an alternative to a particular class (Kadziński & Tervonen, 2013), and
- recommendation obtained with a value function which is representative for the whole set of compatible value functions (Kadziński, Greco, & Słowiński, 2013).

This paper can be seen as an inherent part of the above mentioned trend in Multiple Criteria Decision Aiding with the following threefold aim.

First of all, we introduce the new type of indirect preference information for sorting problems in the form of assignment-based pairwise comparisons of alternatives. Indeed, people are used to refer to such comparisons in their judgments. In many real-world decision situations, they use statements such as “a should be assigned to a class at least as good as b”, “there is a difference of at least two classes between c and d”, “e is better than f by at most one class only”, or “g and h represent the same class”. These are imprecise preference statements, which refer to the desired assignments for pairs of alternatives, but without specifying any concrete class. Note that when using such expressions, people do not rate a given alternative individually as in the assignment examples, but rather confront alternatives “one vs. one”. Nevertheless, the purpose of these statements is not to rank the alternatives, but rather to enable their comparison in terms of the sorting problem.

Furthermore, as mentioned before, the authors of Kadziński and Tervonen (2013) provided procedures for comparing sorting recommendation for pairs of alternatives. Precisely, they introduced the necessary and possible assignment-based preference relations corresponding to such results as, “irrespective of the compatible model instance, the class of alternative a is never worse than the class of b” or “there is at least one compatible model instance that assigns a to a class at least as good as b”. Such a recommendation involving all compatible preference model instances and referring directly to pairs of alternatives is not possible when using the necessary and possible assignments only. Given a framework for comparing pairs of alternatives at the output, it is even more justified to allow providing pairwise comparisons at the input of the method too.

Moreover, specification of the assignment-based pairwise comparisons of the alternatives allows to address one of the commonly acknowledged disadvantages of using some traditional disaggregation methods. Very often, the ranges of possible assignments for the alternatives are rather wide and there exist significant subsets of alternatives possibly assigned to the same class range (see, e.g., Greco, Kadziński, Mousseau, & Słowiński, 2012; Kadziński et al., 2013). Accounting for the assignment-based pairwise comparisons reduces the set of compatible preference model instances, thus making the possible assignments more precise and diversifying the recommendation obtained for different alternatives.

The second aim of the paper is to provide a framework for incorporating a number of preference modeling approaches into a single modeling approach capturing preference information given in different forms. These include assignment examples, assignment-based pairwise comparisons, and desired class cardinalities. However, we additionally account for other types of preference information concerning the shape of the marginal value functions (e.g., concavity or convexity, interval estimates of relative values, and intensities of preference) and newly introduced desired comprehensive values of alternatives assigned to a given class or class range (e.g., “alternatives assigned to class at most medium should have value not greater than 0.4” or “the difference of values between alternatives assigned to class good and bad should be at least 0.7”). We believe that such desired values are easier to provide for the DMs than, e.g., the range of variation of piecewise linear marginal value functions, and may be appreciated by some DMs also from the point of view of interpretability of the results. Accounting for all these preference statements, we provide a flexible modeling framework that incorporates a wide spectrum of indirect and imprecise preference information coming from the DM. Obviously, the ultimate goal of using all this preference information consists in applying the inferred compatible preference model on the whole set of alternatives.

In that respect, the third aim of the paper is to provide a framework for deriving a variety of results stemming from robustness analysis, including necessary and possible assignments, necessary and possible assignment-based preference relations, and extreme class cardinalities. These outputs correspond to different types of admitted preference information, i.e., assignment examples, assignment-based pairwise comparisons, and desired class cardinalities. In this way, the preference information of each type is reproduced in the respective outcome. The DM may also observe the impact of her/his preferences on the sorting recommendation concerning the whole set of alternatives (in case of assignments), all pairs of alternatives (in case of assignment-based preference relations), and all classes (in case of extreme class cardinalities). By exhibiting three types of results we encourage the DM to add some exemplary decisions or requirements that lead to a final recommendation. Let us call this integrated multiple criteria sorting framework with different types of preference information and sorting results, ROR-UTADIS.

The organization of the paper is the following. In the next section, we introduce definitions and notation that will be used along the paper. In Section 3, we recall procedures of multiple criteria sorting which have been used and extended in the proposed approach. Section 4 presents different types of preference information that may be supplied by the DM. This includes the newly introduced assignment-based pairwise comparisons and the desired comprehensive values for alternatives assigned to a given class or class range. Section 5 is devoted to exploitation of the set of compatible preference model instances in terms of class assignments, assignment-based pairwise comparisons, and extreme class cardinalities. In Section 6, we discuss the impact of preference information on recommendation, putting emphasis on incremental specification of preferences. Section 7 demonstrates the approach on an example of its application. The last section concludes the paper.

2. Notation

We shall use the following notation:

- $A = \{a_1, a_2, \ldots, a_i, \ldots, a_n\}$—a finite set of n alternatives;
- $A^R = \{a^*, b^*, \ldots\}$—a finite set of reference alternatives, on which the DM accepts to express preferences; we assume that $A^R \subseteq A$;
- $G = \{g_1, g_2, \ldots, g_j, \ldots, g_m\}$—a finite set of m evaluation criteria, $g_j : A \to \mathbb{R}$ for all $j \in J = \{1, 2, \ldots, m\}$;
- $X_j = \{x_j \in \mathbb{R} : g_j(a_i) = x_j, a_i \in A\}$—the set of all different evaluations on $g_j, j \in J$; we assume, without loss of generality, that the greater $g_j(a_i)$, the better alternative a_i on criterion g_j, for all $j \in J$;
- $x_j^1, x_j^2, \ldots, x_j^{n_j(A)}$—the ordered values of X_j, $x_j^k < x_j^{k+1}, k = 1, 2, \ldots, n_j(A) - 1$, where $n_j(A) = |X_j|$ and $n_j(A) \le n$;
- $g_{j,*}$ and g_j^* are, respectively, the lower and upper bounds for the performance scale on g_j (in particular, if these extreme performances are not predefined, we assume they are equal to, respectively, the worst and the best performances of the existing alternatives, i.e. $g_{j,*} = x_j^1$ and $g_j^* = x_j^{n_j(A)}$);
- $C_1, C_2, \ldots, C_p$—p predefined preference-ordered classes, where C_{h+1} is preferred to C_h, $h = 1, \ldots, p-1$, moreover, $H = \{1, \ldots, p\}$.

In order to represent DM's preferences, we shall use a model in form of an additive value function:

$$U(a) = \sum_{j=1}^{m} u_j(g_j(a)) = \sum_{j=1}^{m} u_j(a). \tag{1}$$

The basic set of constraints defining general additive value functions has the following form:

$$\left.\begin{array}{l} [M_1]\ u_j(x_j^k) - u_j(x_j^{(k-1)}) \ge 0, \quad k = 2, \ldots, n_j(A),\ j = 1, \ldots, m, \\ [M_2]\ u_j(x_j^1) \ge u_j(g_{j,*}), \quad u_j(x_j^{n_j(A)}) \le u_j(g_j^*), \\ [N_1]\ u_j(g_{j,*}) = 0, \quad j = 1, \ldots, m,\ \sum_{j=1}^{m} u_j(g_j^*) = 1. \end{array}\right\} E_{EX}^{BASE}$$

Constraints $[M_1]$ and $[M_2]$ ensures that marginal value functions u_j, $j \in J$, are monotone, non-decreasing, while constraint $[N_1]$ guarantees that the additive value (1) is bounded within the interval $[0, 1]$.

As noted in Spliet and Tervonen (2014), analysis of general monotonic marginal value functions is unlikely to be useful for practical decision aiding in contexts where the decision maker preferences are elicited in the form of holistic judgments. Thus, instead of general marginal value functions we can use piecewise linear ones. Then, for each u_j, $j = 1, \ldots, m$, we need to define the number of characteristic points γ_j. The intervals $[g_{j,*}, g_j^*]$ are divided into $\gamma_j - 1$ equal sub-intervals with the endpoints:

$$g_j^s = g_{j,*} + (g_j^* - g_{j,*})(s-1)/(\gamma_j - 1), \quad s = 1, \ldots, \gamma_j.$$

Then, the monotonicity constraints $[M_1]$ and $[M_2]$ should be formulated in the following way:

$$\left.\begin{array}{l} u_j(g_j^s) - u_j(g_j^{s-1}) \ge 0, \quad j = 1, \ldots, m,\ s = 1, \ldots, \gamma_j, \\ u_j(x_j^k) = u_j(g_j^{s-1}) + (u_j(g_j^s) - u_j(g_j^{s-1}))(x_j^k - g_j^{s-1})/(g_j^s - g_j^{s-1}) \\ \text{ for all } x_j^k \in [g_j^{s-1}, g_j^s], \quad j = 1, \ldots, m,\ k = 1, \ldots, n_j(A). \end{array}\right\} \tag{2}$$

3. Reminder on value-based sorting procedures

When using a value-function U as a preference model, we may employ either a threshold- or an example-based sorting procedure.

3.1. Threshold-based sorting

In the threshold-based sorting procedure, we represent the DM preferences with a pair $(U, \mathbf{t})$, where U is an additive value function and $\mathbf{t}$ is a vector of value thresholds that separate the classes. A vector $\mathbf{t} = \{t_1, \ldots, t_{p-1}\}$ is defined so that $0 < t_1 < \cdots < t_{p-1} < 1$, and t_{h-1} and t_h are, respectively, the lower and upper threshold of class C_h, $h = 2, \ldots, p-1$ (see, e.g., Zopounidis & Doumpos, 2000). Note that t_1 is an upper threshold of class C_1 while the lower threshold is 0, and t_{p-1} is a lower threshold of class C_p while the upper threshold is > 1. Thus, the basic set of constraints is the following:

$$\left.\begin{array}{l} t_1 \ge \varepsilon, \quad t_{p-1} \le 1 - \varepsilon, \\ t_h - t_{h-1} \ge \varepsilon, \quad h = 2, \ldots, p-1, \\ E_{EX}^{BASE}, \end{array}\right\} E_{TH}^{BASE} \tag{3}$$

where ε is a (generally small) positive value.

The threshold-based sorting model is completely defined by $(U, \mathbf{t}) \in (\mathcal{U}, \mathbf{t})^R$, and alternative a is assigned to class C_h ($a \to C_h$) iff $U(a) \in [t_{h-1}, t_h[$.

3.2. Example-based sorting

In the example-based sorting procedure the classes are delimited by the assignment examples provided by the DM. Each assignment example consists of a reference alternative $a^* \in A^R \subseteq A$ and its desired assignment:

$$a^* \to [C_{L^{DM}(a^*)}, C_{R^{DM}(a^*)}], \tag{4}$$

where $[C_{L^{DM}(a^*)}, C_{R^{DM}(a^*)}]$ is an interval of contiguous classes $C_{L^{DM}(a^*)}$, $C_{L^{DM}(a^*)+1}, \ldots, C_{R^{DM}(a^*)}$. An assignment example is said to be precise if $L^{DM}(a^*) = R^{DM}(a^*) = h$ for some $h \in H$ (then let us denote it by $C_{h^{DM}(a^*)}$), and imprecise, otherwise. For the conditions ensuring that a set of assignment examples is reproduced by a value function U, see Section 4.1.2.

Assuming the use of a single value function U in the example-based procedure, alternative a is assigned to an interval of classes $[C_{L^U(a)}, C_{R^U(a)}]$ ($a \to [C_{L^U(a)}, C_{R^U(a)}]$) where:

$$L^U(a) = \mathrm{Max}\{\{1\} \cup \{L^{DM}(a^*) : U(a^*) \le U(a),\ a^* \in A^R\}\}, \tag{5}$$

$$R^U(a) = \mathrm{Min}\{\{p\} \cup \{R^{DM}(a^*) : U(a^*) \ge U(a),\ a^* \in A^R\}\}. \tag{6}$$

The relationships between the threshold- and example-based sorting procedures are studied in Greco et al. (2010) and Kadziński and Tervonen (2013). From a cognitive point of view, the threshold-based procedure is easier to explain to the non-experienced users, because it materializes the class frontiers with the precise thresholds on the scale of a comprehensive value, while in the example-based procedure these boundaries are rather implicit. When it comes to the delivered results, the example-based procedure provides potentially imprecise assignments, while the threshold-based procedure always delivers precise results. Such unambiguity of the delivered recommendation is required in some decision making situations. As proved in Greco et al. (2010), the example-based procedure is more general than the threshold-based one, because it implicitly considers intervals of possible thresholds instead of single values for the thresholds. Precisely, the example-based procedure using a single value function U indicates for each alternative possibly imprecise assignment interval composed of all classes indicated by the threshold-based procedure defined with the same value function U and different admissible vectors of thresholds $\mathbf{t}$. Finally, as shown later throughout the paper, a computational cost of using the example-based procedure is higher than for the threshold-based one. The underlying mathematical models considered here involve more constraints and more binary variables. Summing up, the selection between the two procedures involves a trade-off between their simplicity and arbitrariness; it should be also conditioned by the context of a specific sorting problem as well as the availability of sufficient computational resources.

4. Preference information

Instead of employing a direct procedure for estimating the comprehensive value model (multi-attribute value theory), we use preference disaggregation analysis. In preference disaggregation, the parameters of the preference model are estimated through the analysis of the DM's preference information concerning some reference alternatives, as well as imprecise requirements with respect to the desired recommendation. The problem is then to estimate a value function that is compatible with the preference information provided by the DM.

In this section, we present a variety of indirect, imprecise, and incomplete preference information admitted by the proposed framework. The wide spectrum of the accounted types of preference information guarantees the flexibility of the interactive procedure. In particular, we admit the variety of indirect preference information in the form of exemplary holistic judgments that should be reproduced by the model, and requirements that the DM would like to impose for the expected results. This means that the DM can use assignment examples, assignment-based pairwise comparisons, or desired class cardinalities. The first two concern a subset of reference alternatives that the DM knows relatively well, whereas desired class cardinalities refer indirectly to the whole set of alternatives. Moreover, we discuss other types of preference information concerning some requirements with respect to the comprehensive values attained by the alternatives assigned to a specific class or to the shape of marginal value functions. We discuss the usefulness of each accounted type of preference information, and we present mathematical models which are able to reproduce preferences of the DM, i.e. translate her/his exemplary decisions into parameters of a value function or of a pair composed of a value function and its vector of thresholds. Let us emphasize that the DM is not obliged to specify preference information of each type. (S)he may refer only to the kind of information (s)he feels comfortable with and that (s)he is able to provide for the problem at hand without an excessive cognitive effort.

4.1. Assignment examples

Indirect specification of preferences in the context of multiple criteria sorting has been traditionally identified with assignment examples in the form:[1]

$a^* \rightarrow [C_{L^{\mathrm{DM}}(a^*)}, C_{R^{\mathrm{DM}}(a^*)}]$.

Assignment examples can be seen as a part of the desired final recommendation that need to be reproduced. At the same time, they provide a view on the way the DM makes decisions, which is subsequently incorporated into parameters of the preference model.

4.1.1. Threshold-based sorting

In the threshold-based sorting procedure, in order to assign reference alternatives to their desired classes, it is necessary to estimate the comprehensive additive value model and the value thresholds that separate the classes. The pairs $(\mathcal{U}, \mathbf{t})^R$ compatible with the provided assignment examples need to satisfy the following set of constraints:

$$\left.\begin{array}{l}\text{for all } a^* \in A^R, a^* \rightarrow [C_{L^{\mathrm{DM}}(a^*)}, C_{R^{\mathrm{DM}}(a^*)}]:\\ \ [TAE_1]\ U(a^*) \geq t_{L^{\mathrm{DM}}(a^*)-1},\\ \ [TAE_2]\ U(a^*) + \varepsilon \leq t_{R^{\mathrm{DM}}(a^*)}.\end{array}\right\} E^{AE}_{TH} \quad (7)$$

Constraints $[TAE_1]$ and $[TAE_2]$ guarantee that comprehensive value of a^*, which is assigned by the DM to a class range $[C_{L^{\mathrm{DM}}(a^*)}, C_{R^{\mathrm{DM}}(a^*)}]$ is not worse than the lower threshold of class $C_{L^{\mathrm{DM}}(a^*)}$ and worse than the upper threshold of class $C_{R^{\mathrm{DM}}(a^*)}$, respectively.

4.1.2. Example-based sorting

In the example-based sorting procedure, the preferences disaggregation process only focuses on the estimation of the comprehensive additive value model. The set $\mathcal{U}^R$ of value functions compatible with the DM's assignment examples needs to satisfy the following constraints:

$$\left.\begin{array}{l}\text{for all } a^*, b^* \in A^R, L^{\mathrm{DM}}(a^*) > R^{\mathrm{DM}}(b^*):\\ \ [EAE_1]\ U(a^*) \geq U(b^*) + \varepsilon.\end{array}\right\} E^{AE}_{EX} \quad (8)$$

Constraint $[EAE_1]$ guarantees that the comprehensive value of a^* is higher than the one of b^* in case the DM assigned the former to a class certainly better than the latter, i.e. if the worst allowed class for a^* is preferred to the best allowed class for b^*.

4.2. Assignment-based pairwise comparisons

Assignment examples are holistic judgments which concern the desired recommendations for individual alternatives. On the contrary, assignment-based pairwise comparisons consist of two reference alternatives $(a^*, b^*) \in A^R \times A^R$ and imprecise comparison between their desired assignments. We account for pairwise comparisons in the following forms:

- a^* is better than b^* by at least $k \geq 0$ classes, denoted by $a^* \succ^{\rightarrow}_{\geq k,\mathrm{DM}} b^*$, or equivalently, there is a difference of at least k classes between a^* and b^* (in case $k = 0$, a^* is assigned to a class at least as good as class of b^*, and in case $k = 1$, a^* is assigned to a class better than b^*)[2];
- a^* is better than b^* by at most $l \geq 0$ classes, denoted by $a^* \succ^{\rightarrow}_{\leq l,\mathrm{DM}} b^*$, or equivalently, there is a difference of at most l classes between a^* and b^* (in case $l = 0$, a^* is assigned to a class not better than (at most as good as) the class of b^*).

[1] We will use "DM" in the lower script to distinguish all types of preference information (assignment examples $a^* \rightarrow [C_{L^{\mathrm{DM}}(a^*)}, C_{R^{\mathrm{DM}}(a^*)}]$, assignment-based pairwise comparisons $a^* \succ^{\rightarrow}_{\geq k,\mathrm{DM}} b^*$ or $a^* \succ^{\rightarrow}_{\leq l,\mathrm{DM}} b^*$, and desired class cardinalities $N^{\min}_{h,\mathrm{DM}}$ and $N^{\max}_{h,\mathrm{DM}}$) provided by the DM from the respective results delivered by the approach.

[2] We will use $\rightarrow$ in the upper script to distinguish assignment-based pairwise comparisons $\succ^{\rightarrow}_{\mathrm{DM}}$ and assignment-based preference relations $\succ^{\rightarrow}$ used in the context of sorting problems from their counterparts ($\succ_{\mathrm{DM}}$ and $\succ$, respectively) traditionally used when dealing with ranking and choice problems.

Obviously, for a single pair $(a^*, b^*) \in A^R \times A^R$, the statements $a^* \succ^{\rightarrow}_{\geq k,\mathrm{DM}} b^*$ and $a^* \succ^{\rightarrow}_{\leq k,\mathrm{DM}} b^*$ may be used together to specify that there is a difference of exactly k classes between a^* and b^* (in case $k = 0$, a^* and b^* are assigned to the same class).

4.2.1. Threshold-based sorting

The pairs $(\mathcal{U}, \mathbf{t})^R$ compatible with the provided assignment-based pairwise comparisons $a^* \succ^{\rightarrow}_{\geq k} b^*$ need to satisfy the following set of constraints:

$$\left.\begin{array}{l}
\text{for all } a^*, b^* \in A^R : a^* \succ^{\rightarrow}_{\geq k,\mathrm{DM}} b^* : \\
\quad \text{for } h = 1, \ldots, p-k : \\
\quad\quad [TPCL_1]\ U(a^*) \geq t_{h+k-1} - Mv_h^{PCL}(a^*, b^*), \\
\quad\quad [TPCL_2]\ U(b^*) + \varepsilon \leq t_h + Mv_h^{PCL}(a^*, b^*), \\
\quad [TPCL_3]\ \sum_{h=1}^{p-k} v_h^{PCL}(a^*, b^*) = p-k-1, \\
\quad [TPCL_4]\ v_h^{PCL}(a^*, b^*) \in \{0,1\},\ h = 1, \ldots, p-k.
\end{array}\right\} E_{TH}^{PCL}$$

The statement $a^* \succ^{\rightarrow}_{\geq k,\mathrm{DM}} b^*$ is equivalent to requiring that in case b^* is assigned to class at most C_h (i.e., its comprehensive value is worse than t_h), then a^* is assigned to class at least C_{h+k} (i.e., its comprehensive value is at least as good as t_{h+k-1}). The character of this pairwise comparison is imprecise, i.e., it does not refer directly to any specific classes C_h and C_{h+k}. Thus, there exist $p-k$ different combinations that need to be accounted as possible assignments for b^* and a^*, i.e., respectively, at most C_1 and at least C_{1+k}, at most C_2 and at least C_{2+k}, …, at most C_{p-k} and at least C_p (see constraints $[TPCL_1]$ and $[TPCL_2]$). Let $v_h^{PCL}(a^*, b^*)$, $h = 1, \ldots, p-k$, be a binary variable (see constraint $[TPCL_4]$) such that when being equal to zero, then a^* is assigned to class at least C_{h+k} and b^* is assigned to class at most C_h. Constraint $[TPCL_3]$ guarantees that exactly one variable $v_h^{PCL}(a^*, b^*)$, $h = 1, \ldots, p-k$, is instantiated with zero. This, in turn, implies that one of the above mentioned combinations holds by ensuring that $U(a^*) \geq t_{h+k-1}$ and $U(b^*) < t_h$ for some $h \in \{1, \ldots, p-k\}$.

The assignment-based pairwise comparison $a^* \succ^{\rightarrow}_{\leq l,\mathrm{DM}} b^*$ is translated to the corresponding set of constraints:

$$\left.\begin{array}{l}
\text{for all } a^*, b^* \in A^R : a^* \succ^{\rightarrow}_{\leq l,\mathrm{DM}} b^* : \\
\quad \text{for } h = 0, \ldots, p-l-1 : \\
\quad\quad [TPCU_1]\ U(a^*) + \varepsilon \leq t_{h+l+1} + Mv_h^{PCU}(a^*, b^*), \\
\quad\quad [TPCU_2]\ U(b^*) \geq t_h - Mv_h^{PCU}(a^*, b^*), \\
\quad [TPCU_3]\ \sum_{h=0}^{p-l-1} v_h^{PCU}(a^*, b^*) = p-l-1, \\
\quad [TPCU_4]\ v_h^{PCU}(a^*, b^*) \in \{0,1\}, h = 0, \ldots, p-l-1.
\end{array}\right\} E_{TH}^{PCU}$$

Analogously to $a^* \succ^{\rightarrow}_{\geq k,\mathrm{DM}} b^*$, the statement $a^* \succ^{\rightarrow}_{\leq l,\mathrm{DM}} b^*$ is equivalent to requiring that in case b^* is assigned to class at least C_h (i.e. its comprehensive value is at least as good as t_h), then a^* is assigned to class at most C_{h+l} (i.e. its comprehensive value is worse than t_{h+l+1}). Again, since the character of this pairwise comparison is imprecise, there exist $p-l$ different combinations that need to be accounted as possible assignments for b^* and a^*, i.e., respectively, at least C_1 and at most C_{1+l}, at least C_2 and at most C_{2+l}, …, at least C_{p-l} and at most C_p (see constraints $[TPCU_1]$ and $[TPCU_2]$). Let $v_h^{PCU}(a^*, b^*)$, $h = 0, \ldots, p-l-1$, be a binary variable (see constraint $[TPCU_4]$) such that when being equal to zero, then a^* is assigned to class at most C_{h+l} and b^* is assigned to class at least C_h. Constraint $[TPCU_3]$ guarantees that at least variable $v_h^{PCU}(a^*, b^*)$, $h = 0, \ldots, p-l-1$, is instantiated with zero. This, in turn, implies that one of the above mentioned combinations holds, i.e. $U(a^*) < t_{h+l+1}$ and $U(b^*) \geq t_h$ for some $h \in \{0, \ldots, p-l-1\}$.

4.2.2. Example-based sorting

The set $\mathcal{U}^R$ of value functions compatible with the DM's assignment-based pairwise comparisons in form $a^* \succ^{\rightarrow}_{\geq k,\mathrm{DM}} b^*$ and $a^* \succ^{\rightarrow}_{\leq l,\mathrm{DM}} b^*$ should satisfy the following sets of constraints E_{EX}^{PCL} and E_{EX}^{PCU}, respectively:

$$\left.\begin{array}{l}
\text{for all } a^*, b^* \in A^R : a^* \succ^{\rightarrow}_{\geq k,\mathrm{DM}} b^* : \\
\quad \text{for } h = 1, \ldots, p-k : \\
\quad\quad [EPCL_1]\ U(a^*) + Mv_h^{PCL}(a^*, b^*) \geq U(a) - M(1 - v_h^{PCL}(a)), \\
\quad\quad\quad\quad \text{for all } a \in A^R : L^{\mathrm{DM}}(a) \geq h+k, \\
\quad\quad [EPCL_2]\ \sum_{a \in A^R : L^{\mathrm{DM}}(a) \geq h+k} v_h^{PCL}(a) \geq 1, \\
\quad\quad [EPCL_3]\ v_h^{PCL}(a) \in \{0,1\},\ \text{for } a \in A^R : L^{\mathrm{DM}}(a) \geq h+k, \\
\quad\quad [EPCL_4]\ U(b^*) - Mv_h^{PCL}(a^*, b^*) \leq U(a) + M(1 - v_h^{PCL}(a)), \\
\quad\quad\quad\quad \text{for all } a \in A^R : R^{\mathrm{DM}}(a) \leq h, \\
\quad\quad [EPCL_5]\ \sum_{a \in A^R : R^{\mathrm{DM}}(a) \leq h} v_h^{PCL}(a) \geq 1, \\
\quad\quad [EPCL_6]\ v_h^{PCL}(a) \in \{0,1\},\ \text{for } a \in A^R : R^{\mathrm{DM}}(a) \leq h, \\
\quad [EPCL_7]\ \sum_{h=1}^{p-k} v_h^{PCL}(a^*, b^*) = p-k-1, \\
\quad [EPCL_8]\ v_h^{PCL}(a^*, b^*) \in \{0,1\},\ h = 1, \ldots, p-k,
\end{array}\right\} E_{EX}^{PCL}$$

and

$$\left.\begin{array}{l}
\text{for all } a^*, b^* \in A^R,\ \text{such that } a^* \succ^{\rightarrow}_{\leq l,\mathrm{DM}} b^* : \\
\quad \text{for } h = 1, \ldots, p-l : \\
\quad\quad [EPCU_1]\ U(a^*) - Mv_h^{PCU}(a^*, b^*) \leq U(a) + M(1 - v_h^{PCU}(a)), \\
\quad\quad\quad\quad \text{for all } a \in A^R : R^{\mathrm{DM}}(a) \leq h+l, \\
\quad\quad [EPCU_2]\ \sum_{a \in A^R : R^{\mathrm{DM}}(a) \leq h+l} v_h^{PCU}(a) \geq 1, \\
\quad\quad [EPCU_3]\ v_h^{PCU}(a) \in \{0,1\},\ \text{for } a \in A^R : R^{\mathrm{DM}}(a) \leq h+l, \\
\quad\quad [EPCU_4]\ U(b^*) + Mv_h^{PCU}(a^*, b^*) \geq U(a) - M(1 - v_h^{PCU}(a)), \\
\quad\quad\quad\quad \text{for all } a \in A^R : L^{\mathrm{DM}}(a) \geq h, \\
\quad\quad [EPCU_5]\ \sum_{a \in A^R : L^{\mathrm{DM}}(a) \geq h} v_h^{PCU}(a) \geq 1, \\
\quad\quad [EPCU_6]\ v_h^{PCU}(a) \in \{0,1\},\ \text{for } a \in A^R : L^{\mathrm{DM}}(a) \geq h, \\
\quad [EPCU_7]\ \sum_{h=1}^{p-l} v_h^{PCU}(a^*, b^*) = p-l-1, \\
\quad [EPCU_8]\ v_h^{PCU}(a^*, b^*) \in \{0,1\}, h = 1, \ldots, p-l.
\end{array}\right\} E_{EX}^{PCU}$$

The underlying modeling reasoning is analogous to the case of the threshold-based procedure. However, instead of comparing the comprehensive values of a^* and b^* with the thresholds that separate the classes, they are compared directly with the comprehensive values of reference alternatives assigned by the DM to a particular class range. Thus, for example, when formulating the conditions for placing a^* to class at least C_{h+k}, instead of requiring that $U(a^*) \geq t_{h+k-1}$, we need to guarantee that $U(a^*)$ is not worse than the comprehensive value of at least one reference alternative $a \in A^R$ assigned by the DM to class at least C_{h+k} (see constraints $[EPCL_1]$, $[EPCL_2]$, and $[EPCL_3]$ that correspond to constraint $[TPCL_1]$ in the threshold-based procedure). Precisely, if a binary variable $v_h^{PCL}(a)$ involved in $[EPCL_1]$ is equal to one, then $U(a^*)$ is not less than $U(a)$ for $a \in A^R : R^{\mathrm{DM}}(a) \leq h+l$. Constraint $[EPCL_3]$ guarantees that at least variable $v_h^{PCL}(a)$, $a \in A^R : R^{\mathrm{DM}}(a) \leq h+l$, is instantiated with one.

On the other hand, to ensure that b^* is assigned to class at most C_h, instead of requiring $U(b^*) < t_h$, we need to guarantee that $U(b^*)$ is worse than the comprehensive value of at least one reference alternative $a \in A^R$ assigned by the DM to class at most C_h (see constraints $[EPCL_4]$, $[EPCL_5]$, and $[EPCL_6]$ that correspond to constraint $[TPCL_2]$ in the threshold-based procedure), etc. The use of binary variables $v_h^{PCL}(a)$ in $[EPCL_{4-6}]$ is analogous to their use in $[EPCL_{1-3}]$.

In any case, when formulating the conditions that guarantee that some alternative is assigned to a class either at least or at most C_h, $h \in H$, in the example-based procedure we need to use more constraints as well as more binary variables than in the threshold-based procedure. Finally, the role of the binary variables $v_h^{PCL}(a^*, b^*)$ and $v_h^{PCU}(a^*, b^*)$ in E_{EX}^{PCL} and E_{EX}^{PCU}, respectively, is the same as in E_{TH}^{PCL} and E_{TH}^{PCU}.

4.3. Desired class cardinalities

Apart from providing holistic judgments in the form of assignment examples or assignment-based pairwise comparisons, the DM may impose requirements on the delivered recommendation concerning desired class cardinalities. In many real-world sorting problems it is necessary to take into account requirements with respect to the desired cardinality of classes, e.g., "we need to reject at least 30 applications and not more than 20 others may deserve further consideration", "at most 10 percent of the employees can be provided with the highest incentive package". In general, each statement of this type consists of a class C_h, $h \in H$, and its desired cardinality being equivalent to the minimal $N^{\min}_{h,\mathrm{DM}}$ and maximal $N^{\max}_{h,\mathrm{DM}}$ number of alternatives that can be assigned to this class either precisely or imprecisely, e.g., "at least 10 alternatives can be assigned to class C_1" or "at most 20 alternatives can be assigned to class C_2". Note that these are most often imprecise statements which refer to the lower and upper limits on the number of alternatives to be assigned to a particular class or to unions of some classes.

Let $v_h(a)$, $a \in A$, $h \in H$, be a binary variable such that when being equal to one, then a is assigned to class C_h either precisely or imprecisely. Now, requiring that C_h should contain at least $N^{\min}_{h,\mathrm{DM}}$ and at most $N^{\max}_{h,\mathrm{DM}}$ alternatives, with $N^{\min}_{h,\mathrm{DM}} \le N^{\max}_{h,\mathrm{DM}}$ corresponds to the following set of constraints:

$$\left.\begin{array}{l}[CL]\ \sum_{a\in A} v_h(a) \ge N^{\min}_{h,\mathrm{DM}},\\ [CU]\ \sum_{a\in A} v_h(a) \le N^{\max}_{h,\mathrm{DM}},\\ [CV]\ v_h(a) \in \{0,1\},\ a \in A,\ h \in H.\end{array}\right\} E^{CC}$$

Constraint $[CL]$ ensures that there are at least $N^{\min}_{h,\mathrm{DM}}$ alternatives assigned to C_h, whereas constraint $[CU]$ guarantees that there are at most $N^{\max}_{h,\mathrm{DM}}$ alternatives assigned to C_h. Obviously, we can use either $[CL]$ or $[CU]$ only to model the requirements concerning the lower or upper bound on the cardinality of class C_h, respectively. Moreover, desired class cardinalities may also refer to the frequencies $N^{\text{min-perc}}_{h,\mathrm{DM}}$ and $N^{\text{max-perc}}_{h,\mathrm{DM}}$ of the set of alternatives (e.g., 10 percent, 25 percent, or 50 percent) instead of absolute values. Then, $N^{\min}_{h,\mathrm{DM}}$ and $N^{\max}_{h,\mathrm{DM}}$ are replaced by $\lceil N^{\text{min-perc}}_{h,\mathrm{DM}} \cdot n\rceil$ and $\lfloor N^{\text{max-perc}}_{h,\mathrm{DM}} \cdot n\rfloor$, respectively.

4.3.1. Threshold-based sorting

The constraints which allow taking into account desired class cardinalities for the threshold-based procedure are the following:

$$\left.\begin{array}{l}\text{for all } a \in A,\ h = 1,\ldots,p:\\ \quad [TC_1]\ U(a) \ge t_{h-1} - M\cdot(1 - v_h(a)),\\ \quad [TC_2]\ U(a) + \varepsilon \le t_h + M\cdot(1 - v_h(a)),\\ [TC_3]\ \sum_{h=1}^{p} v_h(a) = 1,\quad \text{for all } a \in A.\end{array}\right\} E^{CC}_{TH}$$

Constraints $[TC_1]$ and $[TC_2]$ compare comprehensive value of a with the lower (t_{h-1}) and the upper (t_h) thresholds of class C_h. If $v_h(a)$ was equal to zero, then this would relax all conditions which are necessary to assign a to class C_h. On the contrary, if $v_h(a)$ was equal to one, then a would be assigned to class C_h, since the corresponding conditions would be satisfied. Constraint $[TC_3]$ admits each alternative to be assigned to a single class only.

4.3.2. Example-based sorting

The constraints which allow taking into account desired class cardinalities for the example-based procedure are the following:

$$\left.\begin{array}{l}\text{for all } a \in A:\\ \quad \text{for all } a^* \in A^R : R^{\mathrm{DM}}(a^*) < h:\\ \quad\ [EC_1]\ U(a) + M\cdot(1 - v_h(a)) \ge U(a^*) + \varepsilon,\ h = 2,\ldots,p,\\ \quad \text{for all } a^* \in A^R : L^{\mathrm{DM}}(a^*) > h:\\ \quad\ [EC_2]\ U(a) + \varepsilon - M\cdot(1 - v_h(a)) \le U(a^*),\ h = 1,\ldots,p-1,\\ [EC_3]\ \sum_{h=1}^{p} v_h(a) \ge 1.\end{array}\right\} E^{CC}_{EX}$$

Constraints $[EC_1]$ and $[EC_2]$ make sure that, when assigned to class C_h (i.e., when $v_h(a) = 1$), the comprehensive value of alternative a is greater than the comprehensive values of reference alternatives assigned by the DM to class at most C_{h-1}, and it is less than the comprehensive values of all reference alternatives assigned by the DM to class at least C_{h+1}. Since the assignment provided by the example-based procedure may be imprecise, we admit each alternative to be possibly assigned to an imprecise class range rather than a single class only (see constraint $[EC_3]$). The latter implies that in case $v_h(a) = 0$, we should explicitly guarantee that a is not assigned to C_h. This requires the use of some additional constraints and binary variables. As a result, modeling desired class cardinalities for an example-based sorting procedure is feasible only for small decision problems. In any case, when formulating the conditions ensuring that some alternative is assigned to class C_h, in the example-based procedure we need to use more constraints than in the threshold-based procedure.

4.4. Additional preference information

In this sub-section, we consider additional types of preference information. The first group of statements we account for concerns the desired comprehensive values or the difference between comprehensive values of alternatives assigned to a given class or class range. These are in agreement with the recent proposal of modeling scores of alternatives attaining some particular rank (Punkka & Salo, 2013). In particular, we consider the following statements and the corresponding constraints:

- The value of an alternative assigned to class at least C_h is at least $u_{\ge h} \in [0,1]$:

  for the threshold-based procedure: $t_{h-1} \ge u_{\ge h}$,
  for the example-based procedure: $U(a^*) \ge u_{\ge h}$,
  for all $a^* \in A^R : L^{\mathrm{DM}}(a^*) \ge h$.

- The value of an alternative assigned to class at most C_h is at most $u_{\le h} \in [0,1]$:

  for the threshold-based procedure: $t_h \le u_{\le h}$,
  for the example-based procedure: $U(a^*) \le u_{\le h}$,
  for all $a^* \in A^R : R^{\mathrm{DM}}(a^*) \le h$.

- The difference of comprehensive values of the alternative assigned to class at least C_h and the alternative assigned to class at most C_k (with $h > k$) is at least $u_{\ge(h,k)}$:

  $$\left.\begin{array}{l}\text{for all } a, b \in A:\\ \quad U(a) + \varepsilon \le t_{h-1} + M(1 - v_{\ge h}(a)),\\ \quad U(b) \ge t_k - M(1 - v_{\le k}(b)),\\ \quad U(a) - U(b) \ge u_{\ge(h,k)} - M(v_{\ge h}(a) + v_{\le k}(b)),\\ \quad v_{\ge h}(a),\ v_{\le k}(b) \in \{0,1\}.\end{array}\right\}$$

  If $U(a) \ge t_{h-1}$ ($U(b) < t_k$), then a binary variable $v_{\ge h}(a)$ ($v_{\le k}(b)$) needs to be equal to one. If both $v_{\ge h}(a)$ and $v_{\le k}(b)$ are equal to one, then the difference between $U(a)$ and $U(b)$ needs to be not less than $u_{\ge(h,k)}$. Thus, the above set of constraints corresponds to the following implication:

  $$\text{if } U(a) \ge t_{h-1} \text{ and } U(b) < t_k \text{ then } U(a) - U(b) \ge u_{\ge(h,k)}.$$

- The difference of comprehensive values of the alternative assigned to class at most C_h and the alternative assigned to class at least C_k (with $h > k$) is at most $u_{\le(h,k)}$:

  $$\left.\begin{array}{l}\text{for all } a, b \in A:\\ \quad U(a) \ge t_h - M(1 - v_{\le h}(a)),\\ \quad U(b) + \varepsilon \le t_{k-1} + M(1 - v_{\ge k}(b)),\\ \quad U(a) - U(b) \le u_{\le(h,k)} + M(v_{\le h}(a) + v_{\ge k}(b)),\\ \quad v_{\le h}(a),\ v_{\ge k}(b) \in \{0,1\}.\end{array}\right\}$$

  The use of binary variables $v_{\le h}(a)$ and $v_{\ge k}(b)$ can be explained analogously to the previous case. Precisely, the above set of constraints corresponds to the following implication:

$$\text{if } U(a) < t_h \text{ and } U(b) \geq t_{k-1} \text{ then } U(a) - U(b) \leq u_{\leq(h,k)}.$$

In case of the example-based procedure, the statements concerning desired difference between comprehensive values of two alternatives are modeled analogously.

The second group of statements concerns the shape of marginal value functions. In particular, we can handle the following statements:

- The concavity of the marginal value function $u_j(a)$ on criterion g_j:

  $$u_j(x_j^i) \leq u_j(x_j^{i-2}) + [u_j(x_j^{i-1}) - u_j(x_j^{i-2})] \times (x_j^i - x_j^{i-2})/(x_j^{i-1} - x_j^{i-2}), \text{ for all } i \in \{3, \ldots, x_j^{n_j(A)}\},$$

  or in case of piece-wise linear function:

  $$u_j(g_j^s) \leq u_j(g_j^{s-2}) + 2[u_j(g_j^{s-1}) - u_j(g_j^{s-2}))], \text{ for all } s \in \{3, \ldots, \gamma_j\}.$$

- The convexity requirement is modeled analogously; then the direction of the above specified inequalities needs to be changed from $\leq$ to $\geq$.
- Ratio estimates of relative values concerning comparison of potential increments, e.g., from x_j^a to x_j^b on criterion g_j (where $x_j^a < x_j^b$) and from x_k^c to x_k^d on criterion g_k (where $x_k^c < x_k^d$):

  $$\alpha[u_k(x_k^c) < u_k(x_k^d)] \leq u_j(x_j^a) - u_j(x_j^b) \leq \beta[u_k(x_k^c) < u_k(x_k^d)],$$

  where α and β are, respectively, lower and upper bounds of the ratio.
- Ordinal value statements (intensities of preference) stating that the importance of the increment from x_j^a to x_j^b on criterion g_j is at least as strong as the importance of the gain from x_k^c to x_k^d on criterion g_k:

  $$u_j(x_j^a) - u_j(x_j^b) \geq u_k(x_k^c) - u_k(x_k^d).$$

In fact, the above statements have been already used in the context of ranking problems (see, e.g., Labreuche, Maudet, Mousseau, & Ouerdane, 2012; Salo & Hämäläinen, 2001), but have not received due attention in view of multiple criteria sorting.

Let us denote the set of constraints stemming from providing by the DM additional types of preference information by E^{ADPI}.

In order to verify that the set of preference model instances (pairs $(\mathcal{U}, \mathbf{t})^R$ or value functions $\mathcal{U}^R$) compatible with preference information provided by the DM is not empty, we consider the following mathematical programming problem:

$$\text{Maximize} : \varepsilon, \ \text{subject to } E^{SORT}, \tag{9}$$

where in case of:

- threshold-based procedure: $E^{SORT} = E_{TH}^{BASE} \cup E_{TH}^{AE} \cup E_{TH}^{PCL} \cup E_{TH}^{PCU} \cup E^{CC} \cup E_{TH}^{CC} \cup E^{ADPI}$;
- example-based procedure: $E^{SORT} = E_{EX}^{BASE} \cup E_{EX}^{AE} \cup E_{EX}^{PCL} \cup E_{EX}^{PCU} \cup E^{CC} \cup E_{EX}^{CC} \cup E^{ADPI}$.

Even though E_{TH}^{BASE} contains additional variables representing the class thresholds when compared with E_{EX}^{BASE}, E^{SORT} for the threshold-based procedure involves fewer constraints and fewer binary variables than its counterpart for the example-based procedure. This is mainly due to the mathematical modeling of assignment-based pairwise comparisons and desired class cardinalities discussed in Sections 4.2 and 4.3.

Let us denote by ε^* the maximal value of ε obtained from the solution of the above Mixed-Integer Linear Programming problem, i.e., $\varepsilon^* = \max \varepsilon$, subject to E^{SORT}. We conclude that $(\mathcal{U}, \mathbf{t})^R$ or $\mathcal{U}^R$ is not empty, if E^{SORT} is feasible and $\varepsilon^* > 0$. In such a case, there exists $\varepsilon > 0$ for which the set of constraints is feasible, which means that all pieces of preference information could be reproduced by at least one value function. On the contrary, when E^{SORT} is infeasible or $\varepsilon^* \leq 0$, some pieces of DM's preference information cannot be reproduced by the assumed preference model and need to be revised.

The general scheme for dealing with incompatibility of the provided preference information with the assumed preference model is based on Mixed-Integer Linear Programming (Mousseau, Dias, & Figueira, 2006). Precisely, with each piece of preference information, we associate a unique binary variable v_{PI}. Then, we rewrite the constraints corresponding to this piece using v_{PI} so that if it was equal to one, the corresponding piece would be eliminated. Let us demonstrate this reformulation for the three basic types of preference information when using the threshold-based procedure. Below, we present only the rewritten constraints, while the rest remains unchanged:

- for each assignment example:

  $$\left.\begin{array}{l} [TAE_1'] \ U(a^*) \geq t_{L^{DM}(a^*)-1} - v_{PI}, \\ [TAE_2'] \ U(a^*) + \varepsilon \leq t_{R^{DM}(a^*)} + v_{PI}; \end{array}\right\}$$

- for each assignment-based pairwise comparison:

  $$\left.\begin{array}{l} [TPCL_3'] \ \sum_{h=1}^{p-k} v_h^{PCL}(a^*, b^*) = p - k - 1 + v_{PI}, \text{ or} \\ [TPCU_3'] \ \sum_{h=0}^{p-l-1} v_h^{PCU}(a^*, b^*) = p - l - 1 + v_{PI}; \end{array}\right\}$$

- for each desired class cardinality:

  $$\left.\begin{array}{l} [CL'] \ \sum_{a \in A} v_h(a) \geq N_{h,\mathrm{DM}}^{\min}(1 - v_{PI}), \\ [CU'] \ \sum_{a \in A} v_h(a) \leq N_{h,\mathrm{DM}}^{\max} + n \cdot v_{PI}. \end{array}\right\}$$

Identifying a minimal subset of troublesome pieces of preference information can be performed by minimizing the sum of binary variables v_{PI}, subject to the rewritten set of constraints $E^{SORT'}$. Obviously, one could try to identify the reasons of inconsistency limiting the search only to a particular kind of preference information (e.g., assignment examples, assignment-based pairwise comparisons, or desired class cardinalities). In this case, we would introduce binary variables only for the constraints related to preferences of this type. Moreover, if the DM is able to express confidence judgments for each piece of preference information, they may be taken into account when inconsistency arises (see Mousseau et al., 2006 for details).

5. Recommendation

Any preference model instance belonging to the set of compatible pairs $(\mathcal{U}, \mathbf{t})^R$ or value functions $\mathcal{U}^R$ reproduces all pieces of preference information given by the DM. The selection of a single model instance fails to investigate whether there are other compatible instances of the preference model which fit the DM's preferences equally well, and it sticks to a rather arbitrary chosen model. In this way, the user is not provided with the possible results in case some other compatible instance of the model or another selection rule was considered. Moreover, in practical decision aiding, where often the preferences need to be co-constructed in an interactive process by the DM and the analyst running the calculations, the DM has the interest in investigating what are the consequences of her/his partial preferences. Thus, in this case providing immediately a final recommendation is not desirable.

Obviously, the sorting recommendation may vary substantially depending on which option of elicited preference information is selected. We apply all compatible preference model instances to work out a recommendation for the set of alternatives A, and examine the influence of variability or imprecision of the input preference information on the variability of the proposed recommendation. In this section, we discuss the wide spectrum of procedures for robustness and sensitivity analysis that could be employed within the proposed framework. By exhibiting results of this analysis, we force the DM to confront her/his value system with the results of applying the inferred model on the set of alternatives. This confrontation leads her/him to gain insights about her/his preferences, providing reactions in the subsequent iteration, as well as to better understand of the employed approach.

For the sake of brevity, we focus on the threshold-based procedure. The suitable algorithms for providing the possible and necessary assignments and assignment-based preference relations are given in Kadziński and Tervonen (2013), while the algorithm for determining the extreme class cardinalities is the same irrespective of the procedure employed.

5.1. Possible and necessary assignments

Given a set of compatible preference model instances, the possible assignment $C_P(a)$ is defined as the set of indices of classes C_h for which there exists at least one compatible model instance assigning a to C_h, and the necessary assignment $C_N(a)$ as the set of indices of classes C_h for which all compatible model instances assign a to C_h.

The possible assignment of $a \in A$ to class C_h, $h \in H$, can be verified by considering the following set of constraints:

$$\left.\begin{array}{l}[TP_1]\;\; U(a) \geq t_{h-1},\;\text{ if } h \geq 2,\\ [TP_2]\;\; U(a)+\varepsilon \leq t_h,\;\text{ if } h \leq p-1,\\ [TP_3]\;\; E^{SORT}.\end{array}\right\} E^{TH}(a \rightarrow^P C_h)$$

Constraints $[TP_1]$ and $[TP_2]$ ensure that the comprehensive value of alternative $a \in A$ is between the lower and upper thresholds of class C_h. Set of constraints $[TP_3]$ defines the set of pairs $(\mathcal{U}, \mathbf{t})^R$ compatible with the DM's preference information. Hence $E^{TH}(a \rightarrow^P C_h)$ has the necessary constraints for assigning a to C_h by the threshold-based procedure for a compatible pair $(U, \mathbf{t})$, i.e. $U(a) \geq t_{h-1}$ and $U(a) < t_h$. We conclude that $\exists (U, \mathbf{t}) \in (\mathcal{U}, \mathbf{t})^R : C^{(U,\mathbf{t})}(a) = h$, i.e. $a \rightarrow^P C_h$, iff $E^{TH}(a \rightarrow^P C_h)$ is feasible and $\varepsilon^* = \max \varepsilon$ s.t. $E^{TH}(a \rightarrow^P C_h) > 0$. Hence $h \in C_P(a)$.

The necessary assignment of $a \in A$ to class C_h, $h \in H$, can be verified by considering the following set of constraints:

$$\left.\begin{array}{l}[TN_1]\;\; U(a)+\varepsilon \leq t_{h-1}+M\cdot v_1,\;\text{ if } h \geq 2,\\ [TN_2]\;\; U(a) \geq t_h - M\cdot v_2,\;\text{ if } h \leq p-1,\\ [TN_3]\;\; v_1+v_2=1,\;\text{ if } 1 \leq h \leq p-1,\\ [TN_4]\;\; v_1, v_2 \in \{0,1\},\\ [TN_5]\;\; E^{SORT}.\end{array}\right\} E^{TH}(a \rightarrow^N C_h)$$

Constraint $[TN_1]$ ensures that in case $v_1 = 0$, the comprehensive value of alternative $a \in A$ is less than the lower threshold of class C_h, i.e. $U(a) < t_{h-1}$. Constraint $[TN_2]$ ensures that in case $v_2 = 0$, the comprehensive value of alternative $a \in A$ is not less than the upper threshold of class C_h, i.e. $U(a) \geq t_h$. Constraints $[TN_3]$ and $[TN_4]$ guarantee that either v_1 or v_2 is equal to zero, i.e., either $U(a) < t_{h-1}$ or $U(a) \geq t_h$ is satisfied. Note that if $h = 1$ $(h = p)$, v_2 (v_1) should be set to zero. Hence $E^{TH}(a \rightarrow^N C_h)$ has the necessary constraints for assigning a to a class different than C_h for a compatible pair $(U, \mathbf{t})$, i.e. either to a class worse than C_h or to a class better than C_h. We conclude that $\forall (U, \mathbf{t}) \in (\mathcal{U}, \mathbf{t})^R : C^{(U,\mathbf{t})}(a) = h$, i.e. $a \rightarrow^N C_h$, iff $E^{TH}(a \rightarrow^N C_h)$ is infeasible or $\varepsilon^* = \max \varepsilon$ s.t. $E^{TH}(a \rightarrow^N C_h) \leq 0$. In this case, there exists no pair $(U, \mathbf{t}) \in (\mathcal{U}, \mathbf{t})^R$ that assigns $a \in A$ to a class worse or better than C_h.

5.2. Necessary and possible assignment-based preference relations

Given a set of compatible preference model instances, the possible assignment-based preference relation $a \succsim^{\rightarrow,P} b$ holds if a is assigned to a class at least as good as class of b (i.e., $a \succsim^{\rightarrow} b$) for at least one compatible model instance, and the necessary assignment-based preference relation $a \succsim^{\rightarrow,N} b$ is true if a is assigned to a class at least as good as class of b for all compatible model instances.

The assignment-based preference relations allow direct comparison of alternatives in terms of their sorting recommendation. This cannot be achieved for all pairs of alternatives when referring only to their necessary and possible assignments. For example, the possible assignments for $a, b \in A$, may be the same, but a may be incomparable with b in terms of $\succsim^{\rightarrow,N}$, or the intersection of the possible assignments for $c, d \in A$, may be non-empty, but c may be assigned to a class strictly better than d for each compatible model instance. The assignment-based preference relations should not be confused with the ranking-specific necessary and possible preference relations introduced in Greco, Mousseau, and Słowiński (2008). The latter ones are defined with respect to the comprehensive values (scores) of alternatives rather than their class assignments. Obviously, the truth of the necessary (possible) preference relation for a pair of alternatives implies also the truth of its assignment-based counterpart. However, the opposite implication is not true. As a result, the necessary and possible assignment-based preference relations need to be verified in a way specific for sorting problems.

The truth of relation $a \succsim^{\rightarrow,P} b$ is verified by considering the following set of constraints:

$$\left.\begin{array}{l}[TPAR_1]\;\; U(a) \geq t_{h-1},\;\text{ if } h \geq 2,\\ [TPAR_2]\;\; U(b)+\varepsilon \leq t_h,\;\text{ if } h \leq p-1,\\ [TPAR_3]\;\; E^{SORT}.\end{array}\right\} E_h^{TH}(a \succsim^{\rightarrow,P} b)$$

Constraint $[TPAR_1]$ ensures that the comprehensive value of alternative $a \in A$ is not less than the lower threshold value of class C_h. Constraint $[TPAR_2]$ ensures that the comprehensive value of alternative $b \in A$ is less than the upper threshold value of class C_h. Hence $E_h^{TH}(a \succsim^{\rightarrow,P} b)$ has the necessary constraints for assigning a to a class at least as good as C_h, and b to a class not better than C_h for a compatible pair $(U, \mathbf{t})$. Thus, it guarantees that a is assigned to a class at least as good as b. We conclude that $a \succsim^{\rightarrow,P} b$ iff $\exists h \in \{1, \ldots, p\}$ such that $E_h^{TH}(a \succsim^{\rightarrow,P} b)$ is feasible and $\varepsilon^* = \max \varepsilon$ s.t. $E_h^{TH}(a \succsim^{\rightarrow,P} b) > 0$.

Relation $a \succsim^{\rightarrow,N} b$ can be computed by considering the following set of constraints:

$$\left.\begin{array}{l}[TNAR_1]\;\; U(b) \geq t_h,\\ [TNAR_2]\;\; U(a)+\varepsilon \leq t_h,\\ [TNAR_3]\;\; E^{SORT}.\end{array}\right\} E_h^{TH}(a \succsim^{\rightarrow,N} b)$$

Constraint $[TNAR_1]$ ensures that the comprehensive value of alternative $b \in A$ is not less than the upper threshold value of class C_h. Constraint $[TNAR_2]$ ensures that the comprehensive value of alternative $a \in A$ is less than the upper threshold value of class C_h. Hence $E_h^{TH}(a \succsim^{\rightarrow,N} b)$ has the necessary constraints for assigning a to a class not better than C_h, and b to a class better than C_h with a compatible pair $(U, \mathbf{t})$. We conclude that $a \succsim^{\rightarrow,N} b$ iff $\forall h \in \{1, \ldots, p-1\}$: $E_h^{TH}(a \succsim^{\rightarrow,N} b)$ given above is infeasible or $\varepsilon^* = \max \varepsilon$ s.t. $E_h^{TH}(a \succsim^{\rightarrow,N} b) \leq 0$. In this case, for all $h \in \{1, \ldots, p-1\}$ there exists no pair $(U, \mathbf{t}) \in (\mathcal{U}, \mathbf{t})^R$ that assigns a to a class at most C_h and b to a class better than C_h.

5.3. Extreme class cardinalities

To compute the minimal $N_h^{\min}$ and maximal $N_h^{\max}$ cardinality of class C_h, the following Mixed-Integer Linear Programming problems need to be solved:

$$\text{Minimize/maximize: } \sum_{a \in A} v_h(a),\; \text{ subject to } E^{SORT}. \tag{10}$$

Let us remind that $v_h(a)$, $a \in A$, $h \in H$, is a binary variable such that when being equal to one, then a is assigned to class C_h either precisely or imprecisely, and when equal to zero, a is assigned to a class or class range not containing C_h. Thus, to obtain the extreme class cardinalities it is sufficient to minimize and maximize the sum of $v_h(a)$ for all $a \in A$, subject to the set of constraints E^{SORT} which define the set of preference model instances compatible with the DM's preference information.

6. Impact of the type of preference information on recommendation

When considering the impact of the type of preference information on the provided results, it is interesting to analyze two aspects. First, let us discuss the correspondence between the provided preference information and the delivered results. For any $a^*, b^* \in A^R$ and $h \in H$, these relations can be summarized as follows:

- $C_N(a^*) \subseteq C_P(a^*) \subseteq [C_{L^{DM}(a^*)}, C_{R^{DM}(a^*)}]$;
- $L^{DM}(a^*) \geq R^{DM}(b^*) \Rightarrow a^* \succsim^{\rightarrow,N} b^*$;
- $a^* \succ^{\rightarrow}_{\geq k,\,DM} b^*,\ k \geq 0 \Rightarrow a^* \succsim^{\rightarrow,N} b^*$;
- $a^* \succ^{\rightarrow}_{\geq k,DM} b^*,\ k > 0 \Rightarrow \text{not}(a^* \succsim^{\rightarrow,P} b^*)$;
- $a^* \succ^{\rightarrow}_{\geq k,DM} b^*,\quad k > 0 \Rightarrow L_P(a^*) \geq L_P(b^*) + k$ and $R_P(a^*) \geq R_P(b^*) + k$;
- $N_h^{min} \geq N_{h,DM}^{min}$ and $N_h^{max} \leq N_{h,DM}^{max}$.

Secondly, let us study the evolution of the results with the growth of preference information. In fact, interactive specification of preference information is encouraged by the proposed method, because presenting the results obtained for the provided preference information, engages the DM in an interaction with the analyst. For example, viewing the possible assignments or extreme class cardinalities, the DM may judge that these ranges are too wide, and supply more precise assignment examples in the next iteration. On the other hand, comparing the necessary and possible assignment-based preference relations, the DM is encouraged to supply preference information that is missing in the necessary relation. The constraints related to every new piece of preference information tend to reduce the feasible polyhedron of all compatible preference model instances. At each iteration $it = 1, \ldots, s$, we can compute the results, and the following interdependencies between outcomes of two subsequent iterations hold:

- $C_{P,it}(a) \subseteq C_{P,it-1}(a)$;
- $\succsim_{it}^{\rightarrow,N}$ and $\succsim_{it}^{\rightarrow,P}$ are nested relations: $\succsim_{it-1}^{\rightarrow,N} \subseteq \succsim_{it}^{\rightarrow,N}$ and $\succsim_{it-1}^{\rightarrow,P} \supseteq \succsim_{it}^{\rightarrow,P}$;
- $[N_{h,it}^{min}, N_{h,it}^{max}] \subseteq [N_{h,it-1}^{min}, N_{h,it-1}^{max}]$.

Thus, with the growth of preference information, the possible assignments and class cardinalities become more precise, the necessary assignment-based preference relation is enriched, while the possible relation is weakened. Obviously, we admit that the DM may remove or modify previously provided pieces of preference information. This is likely to happen, e.g., when the DM changed her/his point of view or in case of inconsistent judgments.

7. Illustrative case study

There is no consensus on how to evaluate cities in terms of liveability. The Economist Intelligence Unit (EIU) proposed to consider the Spatially Adjusted Liveability Index, which is based on the living conditions as well as spatial aspects of city life, such as its urban form, geographical situation, cultural assets, and pollution. Reconsidering the EIU study, we take into account four criteria with an increasing direction of preference: stability (g_1), culture and environment (g_2), infrastructure (g_3), and spatial characteristics (g_4). We focus on 24 cities from Asia with the aim of placing them within one of four classes C_1–C_4, such that C_1 is the worst class and C_4 is the best class (see Table 1 for cities' performances).

We apply the threshold-based sorting procedure and assume a DM to have provided preference information of different types, including:

- six exemplary assignments (see Table 2), e.g., Phnom Penh is assigned to the worst class C_1;
- three assignment-based pairwise comparisons (see Table 3), e.g., Hanoi is claimed to be better than Ho Chi Minh City by one class (i.e., it is better by at least and at most one class at the same time);
- desired class cardinalities for all classes stating that the number of alternatives assigned to each class should be between three and nine (see Table 4).

Table 1
Cities' performances.

	g_1	g_2	g_3	g_4
Hong Kong	95.0	85.9	96.4	75.0
Osaka	90.0	93.5	96.4	64.0
Tokyo	90.0	94.4	92.9	53.3
Seoul	80.0	85.6	89.3	58.8
Singapore	95.0	76.6	100	46.7
Beijing	80.0	72.2	85.7	51.5
Shanghai	80.0	75.0	75.0	46.1
Shenzhen	85.0	63.7	82.1	48.5
Kuala Lumpur	80.0	67.8	76.8	36.6
Tianjin	90.0	65.3	82.1	27.7
Guangzhou	80.0	61.1	76.8	42.9
New Delhi	55.0	55.6	58.9	58.6
Dalian	85.0	62.0	75.0	21.0
Manila	60.0	63.2	75.0	21.0
Bangkok	50.0	64.4	69.6	36.3
Mumbai	60.0	56.3	51.8	52.1
Jakarta	50.0	59.3	57.1	42.3
Hanoi	55.0	53.7	51.8	38.4
Damascus	55.0	54.2	55.4	36.5
Tashkent	50.0	55.3	51.8	26.8
Ho Chi Minh City	55.0	49.5	48.2	35.1
Tehran	50.0	35.9	33.9	53.6
Phnom Penh	60.0	49.3	53.6	24.1
Karachi	20.0	38.7	51.8	48.5

Table 2
Assignment examples.

Class	First iteration	Second iteration	Third iteration
C_1	Phnom Penh		
C_2	New Delhi, Jakarta	Manila	Dalian
C_3	Beijing, Guangzhou	Tianjin	
C_4	Tokyo	Singapore	

These preference statements are consistent and the set of compatible instances of the preference model (value functions and class threshold values) is not empty.

The possible assignments are presented in Table 5. As noted in Kadziński and Tervonen (2013), when using a threshold-based sorting procedure, the necessary assignment is either empty (in case there is no unanimity with respect to the assignment provided by different compatible pairs $(U, \mathbf{t})$) or precise (if possible assignment is precise as well). For brevity, we skip explicit presentation of the necessary assignments. In any case, for six reference cities, the possible and necessary assignments are not empty. Another six cities (Hong Kong, Osaka, Seoul, Mumbai, Hanoi, and Ho Chi Minh City), which have not been referred by the DM as assignment examples are precisely assigned to a single class. For the remaining 12 alternatives, the necessary assignment is empty and the possible assignment is imprecise. There are nine cities possibly assigned to two consecutive classes and three cities with a possible assignment of three classes (Shanghai, Tianjin, and Manila).

A Hasse diagram of the necessary assignment-based preference relation $\succsim^{\rightarrow,N}$ is presented in Fig. 1. When analyzing this diagram, we can distinguish three types of assignment-based relations derived from $\succsim^{\rightarrow,N}$:

- Indifference $\sim^{\rightarrow,N}$ (i.e., a symmetric part of $\succsim^{\rightarrow,N}$), which holds for pairs of cities assigned to the same class by all compatible value functions and class thresholds (e.g., irrespective of the compatible model instance, Hong Kong, Osaka, Tokyo, and Seoul are assigned to C_4). These cities are grouped together within a single node in

Table 3
Assignment-based pairwise comparisons.

Symbols	First iteration	Symbols	Second iteration	Symbols	Third iteration
11	Seoul $\succ^{\rightarrow}_{\geq 2,\mathrm{DM}}$ Mumbai	21	Shanghai $\succ^{\rightarrow}_{\geq 0,\mathrm{DM}}$ Shenzhen	31	Damascus $\succ^{\rightarrow}_{\geq 1,\mathrm{DM}}$ Tashkent
12	Kuala Lumpur $\succ^{\rightarrow}_{\geq 1,\mathrm{DM}}$ Bangkok	21	Shanghai $\succ^{\rightarrow}_{\leq 0,\mathrm{DM}}$ Shenzhen	31	Damascus $\succ^{\rightarrow}_{\leq 1,\mathrm{DM}}$ Tashkent
13	Hanoi $\succ^{\rightarrow}_{\geq 1,\mathrm{DM}}$ Ho Chi Minh City	22	Bangkok $\succ^{\rightarrow}_{\geq 1,\mathrm{DM}}$Tehran		
13	Hanoi $\succ^{\rightarrow}_{\leq 1,\mathrm{DM}}$ Ho Chi Minh City				

Table 4
Desired class cardinalities.

	First iteration		Second iteration		Third iteration	
Class	$N^{\min}_{h,\mathrm{DM}}$	$N^{\max}_{h,\mathrm{DM}}$	$N^{\min}_{h,\mathrm{DM}}$	$N^{\max}_{h,\mathrm{DM}}$	$N^{\min}_{h,\mathrm{DM}}$	$N^{\max}_{h,\mathrm{DM}}$
C_1	3	9	4	8	5	6
C_2	3	9	4	8	7	8
C_3	3	9	4	7	6	7
C_4	3	9	4	7	5	5

Table 5
Possible assignments.

	First iteration	Second iteration	Third iteration
Hong Kong	C_4	C_4	C_4
Osaka	C_4	C_4	C_4
Tokyo	C_4	C_4	C_4
Seoul	C_4	C_4	C_4
Singapore	$C_3 - C_4$	C_4	C_4
Beijing	C_3	C_3	C_3
Shanghai	$C_2 - C_4$	C_3	C_3
Shenzhen	$C_3 - C_4$	C_3	C_3
Kuala Lumpur	$C_2 - C_3$	C_3	C_3
Tianjin	$C_2 - C_4$	C_3	C_3
Guangzhou	C_3	C_3	C_3
New Delhi	C_2	C_2	C_2
Dalian	$C_2 - C_3$	$C_2 - C_3$	C_2
Manila	$C_1 - C_3$	C_2	C_2
Bangkok	$C_1 - C_2$	C_2	C_2
Mumbai	C_2	C_2	C_2
Jakarta	C_2	C_2	C_2
Hanoi	C_2	C_2	C_2
Damascus	$C_1 - C_2$	$C_1 - C_2$	C_2
Tashkent	$C_1 - C_2$	$C_1 - C_2$	C_1
Ho Chi Minh City	C_1	C_1	C_1
Tehran	$C_1 - C_2$	C_1	C_1
Phnom Penh	C_1	C_1	C_1
Karachi	$C_1 - C_2$	$C_1 - C_2$	C_1

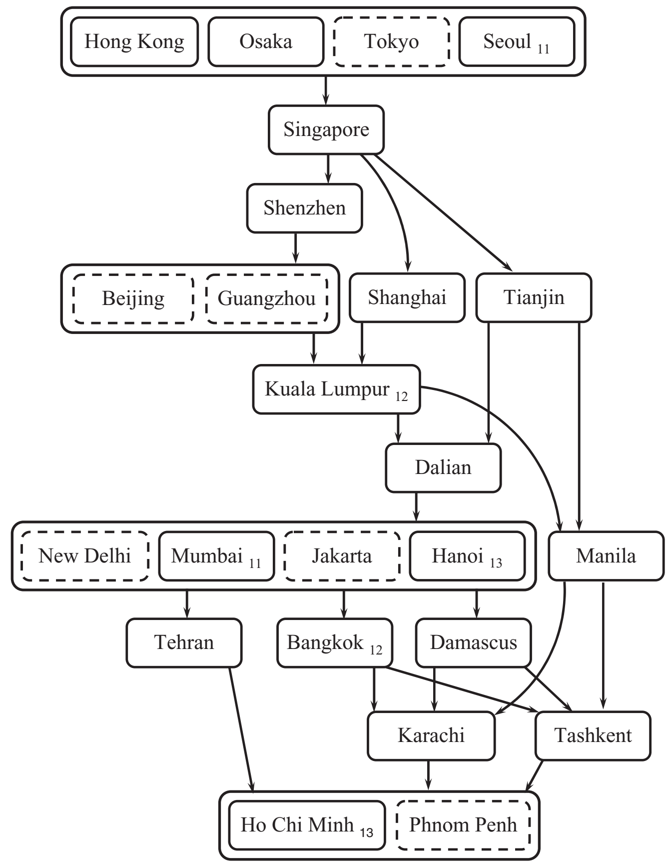

Fig. 1. Hasse diagram of the necessary assignment-based preference relation in the first iteration (dashed line indicates assignment example provided in the current iteration; symbols 11, 12 and 13 distinguish alternatives compared pairwise).

Fig. 1. This interpretation of indifference is consistent with the general definition of decision classes in sorting problems, which is related to the way in which alternatives assigned to each class would be further processed. This treatment needs to be the same for all alternatives placed in the same class.

- Strict preference $\succ^{\rightarrow,N}$ (i.e., an asymmetric part of $\succsim^{\rightarrow,N}$), which holds for pairs of cities $(a, b) \in A \times A$ (e.g., (Osaka, Singapore) and (Osaka, Shanghai)), such that a is assigned to a class at least as good as b by all compatible value functions and class thresholds, while the inverse relation does not hold (i.e., *not* $(b \succsim^{\rightarrow,N} a)$, which means that there is at least one compatible model instance that assigns b to a class worse than a). These pairs of cities are connected by an arrow in Fig. 1 (since $\succ^{\rightarrow,N}$ is transitive, the arcs obtainable by the transitive closure are omitted in the figure).
- Incomparability $R^{\rightarrow,N}$, which holds for pairs of cities $a, b \in A$ (e.g., (Shenzhen, Shanghai), (Hanoi, Manila)) which are not related by the necessary relation $\succsim^{\rightarrow,N}$. This means that for some compatible model instances a is assigned to a class strictly better than b, whereas for some other compatible instances the order of classes is inverse. These cities are not related by an arc in Fig. 1 (neither directly nor when considering transitivity of the necessary relation).

As noted in Kadziński and Tervonen (2013) the purpose of the assignment-based relations is not to rank the alternatives, but merely to enable their pairwise comparisons in a manner compatible with the sorting method.

Obviously, $\succsim^{\rightarrow,N}$ reproduces the assignment-based pairwise comparisons provided by the DM (e.g., Seoul $\succ^{\rightarrow}_{\geq 2,\mathrm{DM}}$ Mumbai $\Rightarrow$ Seoul $\succ^{\rightarrow,N}$ Mumbai). Relation $\succsim^{\rightarrow,N}$ is also consistent with the provided assignment examples. Alternatives assigned by the DM to a class strictly better are necessarily preferred to those assigned to a class worse (see, e.g., (Tokyo, Beijing), (Guangzhou, Phnom Penh)). Moreover, alternatives assigned by the DM to the same class are indifferent in terms of $\succsim^{\rightarrow,N}$ (see, e.g., (Beijing, Gaungzhou), (New Delhi, Jakarta)). Relation $\succsim^{\rightarrow,N}$ reveals interdependencies between class assignments for pairs of alternatives which are not obvious when considering only the possible assignments. For example, even though Singapore and Shenzhen are both possibly assigned to classes C_3 and C_4, Singapore is always placed in a class at least as good as Shenzhen, while the opposite does not hold. Such information is particularly useful when the possible assignments of alternatives are imprecise. Finally, let us refer to the impact of assignment-based pairwise comparisons on the possible assignments. For example, Seoul which has

Table 6
Class cardinalities.

Class	First iteration N_h^{min}	First iteration N_h^{max}	Second iteration N_h^{min}	Second iteration N_h^{max}	Third iteration N_h^{min}	Third iteration N_h^{max}
C_1	3	8	4	6	5	5
C_2	4	9	6	8	8	8
C_3	3	9	6	7	6	6
C_4	4	7	5	5	5	5

been judged to be better than Mumbai by at least two classes is necessarily assigned to C_4, while Mumbai is assigned to C_2 with all compatible preference model instances. On the other hand, while C_P(Kuala Lumpur) $= [C_2, C_3]$, C_P(Bangkok) $= [C_1, C_2]$, and the decision maker required Kuala Lumpur $\succ^{\rightarrow}_{\geq 1,\mathrm{DM}}$ Bangkok, we know that when Kuala Lumpur is assigned to C_3 or C_2, Bangkok is assigned, respectively, to C_2 or C_1 (they are never placed together in C_2).

When it comes to the extreme class cardinalities (see Table 6), they respect the desired cardinalities provided by the DM. For three classes (C_1, C_2, and C_4), the cardinality interval is more precise than the one imposed by the DM. For example, there are between four and seven cities assigned to the best class, while the allowed limits are between three and nine. It is also interesting to collate these class cardinalities with possible assignments. For example, while there are eight cities possibly assigned to C_4, up to seven are placed in this class at the same time. In the same spirit, while there are only two cities necessarily assigned to C_1, at least three cities are always placed in this class by any compatible preference model instance.

The above analysis reveals the multilayer interdependencies between different types of preference information and sorting results. What is more, these results stimulate the DM to enrich her/his preferences in the next iteration. Let us suppose that considering the results of the first iteration, the DM feels confident that:

- another three cities (Manila, Tianjin, and Singapore) with imprecise possible assignments should be assigned precisely to a single class (see Table 2);
- Shenzhen and Shanghai should be assigned to the same class, while Bangkok should be assigned to a class better than Tehran (see Table 3); these two pairs of alternatives were incomparable in terms of the necessary assignment-based preference relation in the first iteration;
- desired class cardinalities should be even more strict than the resulting cardinality intervals after the first iteration (see Table 4).

When compared to the previous iteration, there are another eight cities (e.g., Shanghai, Bangkok, and Tehran) with a non-empty precise possible and necessary assignments (see Table 5). For only four cities there is a hesitation with respect to their assignment.

Further, the possible and necessary assignment-based preference relations converge with the growth of preference information (see Fig. 2). In particular, the necessary partial preorder is enriched (e.g., Dalian $\succsim^{\rightarrow,N}$ Manila, while this was not true in the previous iteration), and the possible relation is weakened. At this stage, all pairs of cities, except one, are related by $\succsim^{\rightarrow,N}$. The cities grouped together in a single node (e.g., New Delhi, Manila, Bangkok, Mumbai, Jakarta, and Hanoi) are assigned to the same class by all compatible preference model instances. Pairs of cities connected by an arrow are related by $\succ^{\rightarrow,N}$. For example, Hong Kong is assigned to a class better than Beijing by all compatible model instances (C_4 vs. C_3). Further, Damascus is assigned to a class at least as good as Karachi for all compatible model instances, while being assigned to a class strictly better for some instances. This means that whenever Karachi is assigned to C_2, the same holds for Damascus, but there exists at least one compatible model instance assigning Karachi to C_1 and Damascus to C_2. The only pair of cities which is incomparable in terms of $\succsim^{\rightarrow,N}$ is (Karachi, Tashkent). Since these cities are possibly assigned to $[C_1, C_2]$, this

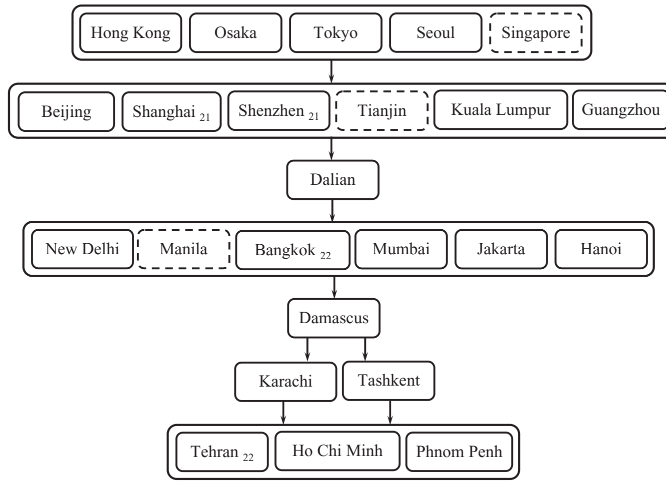

Fig. 2. Hasse diagram of the necessary assignment-based preference relation in the second iteration.

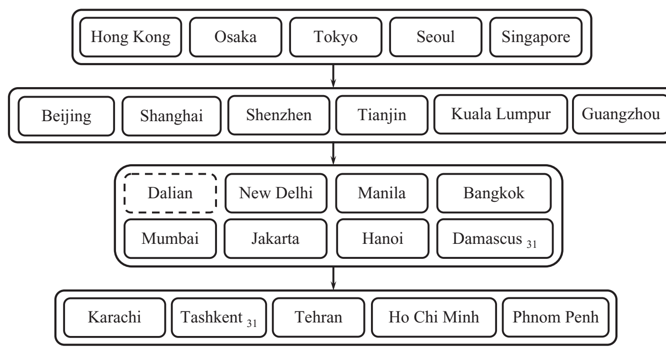

Fig. 3. Hasse diagram of the necessary assignment-based preference relation in the third iteration.

incomparability means that with some compatible model instances Karachi is placed in C_2 and Taskhent is assigned to C_1, while for some other compatible model instances, the order of classes is inverse. Note that the last two observations cannot be derived directly from the necessary and possible assignments.

Finally, the range of class cardinalities in the second iteration is narrower for all classes (see Table 6). For example, all compatible preference model instances assign five alternatives to class C_4. In general, incremental specification of preference information allows obtaining more precise recommendation. It is the case, since new pieces of preference information constrained the set of compatible value functions and class thresholds.

Obviously, the interactive process can be pursued until the obtained results are decisive enough for the DM. Let us assume that in the third iteration, the DM provides additional exemplary assignment (see Table 2), another assignment-based pairwise comparison (see Table 3), and more precise desired class cardinalities (see Table 4). As a result, the possible and necessary assignments for all cities are precise (see Table 5), the cardinalities of all classes are exact (see Table 6), and the necessary assignment-based preference relation is a complete pre-order. When analyzing the Hasse diagram of this relation presented in Fig. 3, we can distinguish four indifference classes composed of cities which are assigned to, respectively, C_4, C_3, C_2, or C_1, by all compatible model instances. Further, all cities assigned to C_4 are related by $\succ^{\rightarrow,N}$ with the cities assigned to C_1, C_2, and C_3, etc. Note, however, that obtaining such a complete pre-order at the end of the decision aiding process is not the target per se. It should be rather seen a side effect of all cities being precisely assigned to some class.

8. Conclusions

In this paper, we presented an integrated preference modeling framework for value-driven multiple criteria sorting, called ROR-UTADIS. We considered a set of preference model instances compatible with assignment examples, desired class cardinalities, and assignment-based pairwise comparisons. The main motivation for introducing the latter type of preference information comes from the common use of such statements in practical situations and some undesired properties of the recommendation worked out with the use of traditional methods. We proposed some Mixed-Integer Programming models which allow the consideration of such preference statements in a preference disaggregation framework. We have also discussed the usefulness of other types of preference information, referring to the shape of marginal value functions or desired comprehensive values of alternatives. Then, we employed example- and threshold-based procedures for deriving sorting recommendation with all compatible preference model instances. This is materialized with the necessary and possible assignments and assignment-based preference relations as well as extreme class cardinalities. We emphasized how decision aiding can benefit from a complementary use of different types of preference information and sorting results within an interactive process.

We envisage the following future developments:

- implementation of a dedicated decision support system within the *diviz* platform (Meyer & Bigaret, 2012); this system will be composed of some modular components so that the users can easily choose the types of preference information and sorting results that they would like to consider;
- thorough experimental analysis of the introduced approach and its application to some real world problems, e.g., in the field of finance;
- adaptation of the integrated framework for multiple criteria sorting to outranking- (Kadziński, Tervonen, & Figueira, 2014) and rule-based (Kadziński, Greco, & Słowiński, 2014) methods and group decision problems;
- extending stochastic ordinal regression (Kadziński & Tervonen, 2013) as well as procedures for computation of preferential reducts (Kadziński, Corrente, Greco, & Słowiński, 2014) to various types of preference information and sorting results proposed in this paper.

Acknowledgments

The first and the second authors acknowledge support from the Polish National Science Centre (grant SONATA, no. DEC-2013/11/D/ST6/03056). The authors thank three anonymous referees for their remarks which helped us to improve the paper.

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